التمرين الثاني

(1

نضع

$u(x) = \ln x$ $\quad u'(x) = \frac{1}{x}$

$v(x) = -\frac{1}{x}$ $\quad v'(x) = \frac{1}{x^2}$

إذن :

$$\begin{aligned} I(a) &= \int_1^a \frac{\ln x}{x^2} dx \\ &= \left[-\frac{1}{x} \ln x \right]_1^a - \int_1^a -\frac{1}{x^2} dx \\ &= \left[-\frac{1}{x} \ln x - \frac{1}{x} \right]_1^a \\ &= \frac{-\ln a - 1}{a} + 1 \end{aligned}$$

(2

$$\begin{aligned} \lim_{a \to +\infty} I(a) &= \lim_{a \to +\infty} \frac{-\ln a - 1}{a} + 1 \\ &= \lim_{a \to +\infty} -\frac{\ln a}{a} - \frac{1}{a} + 1 \\ &= 1 \end{aligned}$$

(3

بما أن لكل $x \geq 1$ ، $x^2 \leq x^2 + 1 \leq 2x^2$ فإن $\frac{1}{2x^2} \leq \frac{1}{x^2+1} \leq \frac{1}{x^2}$. وبما أن $\ln x \geq 0$ $(x \geq 1)$ فإن $\frac{\ln x}{2x^2} \leq \frac{\ln x}{x^2+1} \leq \frac{\ln x}{x^2}$ و منه فإن $\frac{1}{2}\int_1^a \frac{\ln x}{x^2} dx \leq \int_1^a \frac{\ln x}{x^2+1} dx \leq \int_1^a \frac{\ln x}{x^2} dx$ أي $\frac{1}{2} I(a) \leq J(a) \leq I(a)$

التمرين الثالث

1) بما أن الدالة $x \mapsto x + \sqrt{x^2 - 1}$ موجبة قطعا و قابلة للإشتقاق على المجال $[1, +\infty[$ فإن الدالة f قابلة للإشتقاق على المجال $[1, +\infty[$ و لكل x من $[1, +\infty[$ لدينا :

$$\begin{aligned} f'(x) &= \frac{1 + \frac{2x}{2\sqrt{x^2-1}}}{x + \sqrt{x^2-1}} \\ &= \frac{\sqrt{x^2+1} + 1}{\sqrt{x^2-1}\left(x + \sqrt{x^2-1}\right)} \\ &= \frac{1}{\sqrt{x^2-1}} \end{aligned}$$

(2

$$K=\int_{\sqrt{2}}^{2}\frac{dx}{\sqrt{x^2-1}}$$
$$=\left[\ln\left(x+\sqrt{x^2-1}\right)\right]_{\sqrt{2}}^{2}$$
$$=\ln\left(2+\sqrt{3}\right)-\ln(\sqrt{2}+1)$$
$$=\ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)$$

التمرين الرابع

(1

$$I_1=\int_0^1\frac{e^x}{1+e^x}dx$$
$$=\left[\ln\left(1+e^x\right)\right]_0^1$$
$$=\ln(1+e)-\ln(2)$$
$$=\ln\left(\frac{1+e}{2}\right)$$

$$I_0+I_1=\int_0^1\frac{dx}{1+e^x}+\int_0^1\frac{e^x}{1+e^x}dx$$
$$=\int_0^1\frac{1+e^x}{1+e^x}dx$$
$$=\int_0^1 1dx$$
$$=[x]_0^1$$
$$=1$$

$$I_n+I_{n+1}=\int_0^1\frac{e^{nx}}{1+e^x}dx+\int_0^1\frac{e^{(n+1)x}}{1+e^x}dx$$
$$=\int_0^1\frac{e^{nx}\left(1+e^x\right)}{1+e^x}dx$$
$$=\int_0^1 e^{nx}dx$$
$$=\frac{1}{n}\int_0^1 ne^{nx}dx$$
$$=\frac{1}{n}\left[e^{nx}\right]_0^1$$
$$=\frac{1}{n}\left(e^n-1\right)$$

- من العلاقة $I_0+I_1=1$ نستنتج أن $I_0=1-I_1$ أي $I_0=1-\ln\left(\frac{1+e}{2}\right)$

2) لكل n من $\mathbb{N}$ و لكل x من $[0;1]$ لدينا $nx\leq(n+1)x$ و بما أن الدالة $x\mapsto e^x$ تزايدية قطعا على $\mathbb{R}$ فإن : $e^{nx}\leq e^{(n+1)x}$ إذن $\frac{e^{nx}}{1+e^x}\leq\frac{e^{(n+1)x}}{1+e^x}$ أي $\int_0^1\frac{e^{nx}}{1+e^x}dx\leq\int_0^1\frac{e^{(n+1)x}}{1+e^x}dx$ مما يقودنا إلى $I_n\leq I_{n+1}$ لكل n من $\mathbb{N}$ أي المتتالية (I_n) تزايدية .

3) لكل x من $[0;1]$ لدينا :

$$\begin{aligned}0\le x\le 1 &\Rightarrow 1\le e^{x}\le e\\ &\Rightarrow 2\le e^{x}+1\le e+1\\ &\Rightarrow \frac{1}{e+1}\le\frac{1}{e^{x}+1}\le\frac{1}{2}\\ &\Rightarrow \frac{e^{nx}}{e+1}\le\frac{e^{nx}}{e^{x}+1}\le\frac{e^{nx}}{2}\end{aligned}$$

ومنه فإن :

$$\int_0^1\frac{e^{nx}}{e+1}dx\le\int_0^1\frac{e^{nx}}{e^{x}+1}dx\le\int_0^1\frac{e^{nx}}{2}dx$$ أي $$\int_0^1\frac{e^{nx}}{e+1}dx\le I_n\le\int_0^1\frac{e^{nx}}{2}dx$$

نعلم أن :

$$\begin{aligned}\int_0^1\frac{e^{nx}}{e+1}dx&=\frac{1}{n(e+1)}\int_0^1 ne^{nx}dx\\&=\frac{1}{n(e+1)}\left[e^{nx}\right]_0^1\\&=\frac{e^{n}-1}{n(e+1)}\end{aligned}$$

$$\begin{aligned}\int_0^1\frac{e^{nx}}{2}dx&=\frac{1}{2n}\int_0^1 ne^{nx}dx\\&=\frac{1}{2n}\left[e^{nx}\right]_0^1\\&=\frac{e^{n}-1}{2n}\end{aligned}$$

إذن $\frac{e^{n}-1}{n(e+1)}\le I_n\le\frac{e^{n}-1}{2n}$.

وبما أن :

$$\begin{aligned}\lim_{n\to+\infty}\frac{e^{n}-1}{n(e+1)}&=\lim_{n\to+\infty}\frac{e^{n}\left(1-\frac{1}{e^{n}}\right)}{n(e+1)}\\&=\lim_{n\to+\infty}\frac{e^{n}}{n}\times\frac{\left(1-\frac{1}{e^{n}}\right)}{(e+1)}\\&=+\infty\end{aligned}$$

$$\begin{aligned}\lim_{x\to+\infty}\frac{e^{n}-1}{2n}&=\lim_{x\to+\infty}\frac{e^{n}\left(1-\frac{1}{e^{n}}\right)}{2n}\\&=\lim_{x\to+\infty}\frac{1}{2}\times\frac{e^{n}}{n}\times\left(1-\frac{1}{e^{n}}\right)\\&=+\infty\end{aligned}$$

فإن : $\lim_{x\to+\infty}I_n=+\infty$ أي (I_n) متتالية متباعدة .

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